Numerical relations between sounds

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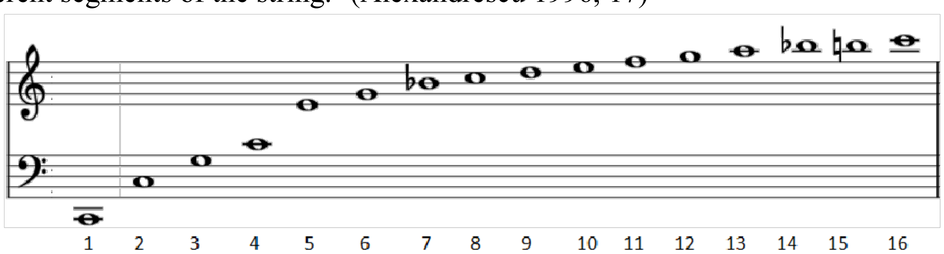
Abstract*: The numbers have been always and are still a fascinating subject. Due to the abstract nature, they have sometimes been associated with mystical elements, but have often contributed to defining more things of an exact value. The first aspect was further reinforced by the fact that certain geometric relationships (such as the square diagonal, the ratio of the length and diameter of the circle or the equal ratio encountered in the pentagram) failed to be defined by whole numbers, keeping the hidden continuum an absolute numerical value. Since ancient times, thinkers have tried to solve these hidden numerical relationships by correlating them with other elements to identify and define them in ways other than numerical values. Therefore, appealing to artistic elements through drawing, architecture, sculpture and music on the other hand was more than necessary to identify means by which to demonstrate the specific type of these relationships.*

Keywords: *Numbers; sounds; mathematical relationships.*

Natural resonance numbers, ratios

Any string vibrates in both its totality and segments. Each of the segment vibrations produces a certain frequency that corresponds to a certain sound pitch. This phenomenon of producing these frequencies is called natural resonance. Dragoş Alexandrescu defines the phenomenon of natural resonance as "fundamental vibrational movement - obtained by the vibration of the whole string, and some partial vibrational movements obtained by vibrating the different segments of the string." (Alexandrescu 1996, 17)



These sounds of natural resonance are called harmonics and occur according to a mathematical progression of type: 1 + 1/2; 1 + 1/3; 1 + 1/4; 1 + 1/5; 1 + 1/6... 1 + 1 / n. The result of this progression in fractions has the following values: 1/1; 3/2; 4/3; 5/4; 6/5; 7/6. In decimal numbers this can be expressed as: 1; 1.5; 1.33; 1.25; 1.20; 1,166 etc.

Overtones, Multiples of prime numbers

The natural resonance sound heard at the perfect octave over the fundamental is the harmonic 2, 4, 8, 16, 32, 64... values that indicate multiples of the number 2, in other words, powers of 2 (2^1; 2^2; 2^3; 2^4; 2^5; 2^6…). Thus, a succession of sounds in octaves indicates the frequency multiplication by 2. In addition, any occurrence at octave of any sound in the harmonic series can be expressed numerically by multiplying by 2.

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The perfect fifth is the third harmonic, and a succession of frequencies from the fifth to fifth is similar to a multiplication by three of the original frequency (fifth – 3; the fifth of fifth - 9, third fifth - 27, fourth fifth - 81, etc.) This multiplication can also be imagined as a succession of powers of the number 3 (3^1; 3^2; 3^3; 3^4; 3^5…). This succession was the basis for calculating the value of sounds in the Pythagorean musical scale.

The major third is the 5th harmonic, and a third-order succession can also be regarded as a succession of powers of number 5. The major third rest on the 5 harmonic (5^1), and the third of third of the 25th (5^2), etc.

Expression of overtone on the basis of the number raised to a power also indicate that all these sounds come from the fundamental sound with the numerical value 1 which is therefore the basis of all the numbers raised at zero power ($2^0 = 3^0 = 5^0 = 1$).

In addition to the prime numbers (numbers divided by 1 and themselves), all the other numbers (composite) are multiples of the primes and, in the case of harmonics, repetitions of the sound they originally expressed. For example here is a short list with the overtones that appear on prime numbers, thus establishing correspondence between it and the musical intervals as shown in natural resonance:

- 2 - perfect octave-8p2/12+ 100%
- 3 – perfect fifth -5p3/21,5+ 50%
- 5 – major third-3M5/41,25+ 25%
- 7 – minor seven -7m7/41,75+75%

Certain ratios that are common as musical intervals are not found in the array of harmonics of a fundamental sound, but result from the correlation of the value between overtones. An example is provided by the perfect fourth that does not appear in the sequence of the first harmonics, but its value is deduced by the ratio of fifth and octave of the fundamental sound (which may justify its treatment as dissonance sometimes).

- 4/3perfect fourth-4p 8/6 (1,333) + 33%
- 9 major second-2M 9/8 (1,125) +12,5%
- 16/15 minor second- 2m 16/15(1,066) +0,66%
- 6/5minor third-3m6/5(1,2) +20%

Mathematical operations with sound

A correlation can be made between the harmonic index and the musical interval that it represents. Thus, multiplying or simplifying any sound in the Harmonic series by 2, 3, 5, etc. may indicate the addition (or decrease) of a musical range corresponding to the value indicated by it. If the frequency of the harmonic 3 is multiplied by 2, then resulting in a frequency that involves the addition of an perfect octave over the perfect fifth. For example, the harmonic 14 is only the octave of the harmonic 7 (14 = 7 * 2) or the harmonic 9 is the perfect fifth of fifth (9 = 3 * 3).

Thus, if we want to add two musical intervals, it is enough to multiply the frequency of a sound with a ratio represented by that interval, or multiply the two fractions between them:

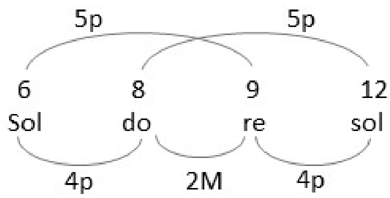
- 3M + 5p = 7M 5/4 * 3/2 = 15/8 or (5 * 3 = 15) - 15 _ 7M
- 3m + 3M = 5p 5/4 * 6/5 = 6/4or (1,2 * 1,25 = 1,5) _ 5p

For mathematical operations we can use both the ratio created between numbers that can indicate a specific interval (example 1.5 = 5p) and the sound result, the corresponding musical interval of the ratio that comes from the natural resonance (12/8 G / C = 5p).

Proportions

The harmonics 6; 8; 9; 12 also hide an interesting mathematical relationship, based on ratio that actually indicate several types of proportions. Matila Ghika evokes in the *Aesthetics and Art Theory* the three types of proportions that the ancients have shown in connection with this series of numbers for which they cite the "universal proportion" of the Pythagoreans (Ghika 1981, 376).

- arithmetic ratio – 6, 9, 12, - where 9 is the arithmetic mean of 6 to 12
- geometric ratio - 12/8 = 9/6
- Harmonic proportion - (12-8)/ (8/6) = 12/6 – where 8 is the harmonic mean between 12 and 6.



$$\frac{do}{sol} = \frac{sol}{re} \qquad \frac{8}{6} = \frac{12}{9} = \frac{4}{3}$$

$$\frac{re}{sol} = \frac{sol}{do} \qquad \frac{9}{6} = \frac{12}{8} = \frac{3}{2}$$

Formulas for the two ratios are:

- Arithmetic mean: (a + b) / 2 = 18/2 = 9,
- Harmonic mean: (2 a x b)/ (a + b) 144/18 = 8.

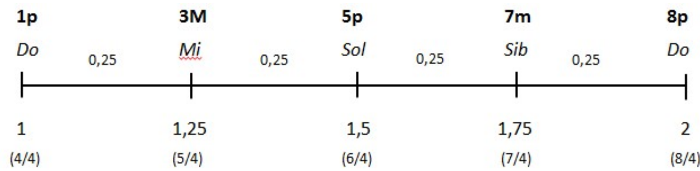
A suggestive example of these values is the cube that has 6 faces, 12 sides and 8 angles. Following the relationship of the 6-8-9-12 harmonics, we can say that it is a balanced and constant one. From this point starts the tuning of instruments like the *lyre* that indicates a fundamental, perfect fourth, the perfect fifth and the perfect octave. At the same time, there are known some brass instruments without valves or other pitch altering devices that emitted only sounds of harmonic series. These frequencies are also noticeable in some melodic lines such as the soldiers' gathering signal (Urmă 1982, 309).

Complementary relationships

From the first sounds from the harmonic series we can see that the relationship between complementary intervals (for example: fifth and fourth) is complementary in relation to frequency and length of string: the fifth - ratio is 1.5 and 1.33 the string while the fourth has a ratio of 1.33 and a string length is 1.5. We can thus say that the relationship between complementary intervals is reversed as a ratio and string length for any pair of sounds of a complementary interval.

Mathematical ratios

Although the arithmetic progression is based on a gradual decrease in which each term is less than its predecessor (2/1 > 3/2 > 4/3 > 5/4 > 6/5 > 7/6… with the values of 2, 1.5, 1.33, 1.25, 1.2, 1.166, 1.142), however, the string lengths corresponding to the fractions placed in ascending order indicate a symmetric distance ratio between sounds arranged in order of pitch. Perfect fifths is exactly halfway between fundamentally sound and perfect octave, the major third midway of perfect fifth interval. Minor seventh is on the halfway between perfect fifth and perfect octave:



This example indicates a balance of proportions of the elements major cord which is one of the most common used chord structures. In addition, this perspective brings a new image of the relationship between the major third and the perfect fifth, embodied here as the point that marks half the distance between the fundamental and the perfect fifth.

In music theory the relationship of the major third within the major chord is studied as a point placed at a distance that unequally divides the seven semitones of the perfect fifth in 4 and 3 halftones (3M + 3m). To find out the value of the fifth we will multiply the frequency ratios between these sounds, two fractions that are uneven 5/4 * 6/5 = 6/4. As a result of summing, however, the major third (5/4) is placed halfway between fundamental (4/4) and perfect fifth (6/4). This indicates the same result for the two different calculation formulas:

$$\frac{5}{4} \cdot \frac{6}{5} = \frac{6}{4} \quad ; \quad 1.2 \times 1.25 = 1.5 \qquad\qquad \frac{4}{4} + \frac{1}{4} = \frac{5}{4} \quad ; \quad \frac{5}{4} + \frac{1}{4} = \frac{6}{4}$$

The equivalence of the ratio found between the first harmonic sounds indicates that in the seventh chord all the sounds are at equal distances (1.25). Going forward, we can see that in the diatonic scale resulting from natural resonance all distances between the intervals are the result of equal ratios, each sound being found on an equal distance to the previous one (1,125).

	+12,5%	**+25%**	**+37,5**	**+50%**	**+62,5%**	**+75%**	**+87,5%**	**+100%**	
1	1,125	1,25	1,375	1,5	1,625	1,75	1,875	2,00	
4/4		5/4		6/4		7/4		8/4	+0,250
8/8	9/8	10/8	11/8	12/8	13/8	14/8	15/8	16/8	+0,125
1p	**2M**	**3M**	**4(+)**	**5p**	**6m (-)**	**7m**	**7M**	**8p**	

Concerning the major chord, can be observed that this is the result of the first harmonics, and the numbers corresponds to the series of Fibonacci (1, 2, 3, 5, 8). In addition, the harmonics 10, 12, 15 and 12, 15, 20 represent series of numbers indicating *harmonic proportions* because they satisfy the formula: (b - a) / a = (c - b) / c; or (c - b) / (b - a) = c / a (Ghyka 1981, 366).

Types of tuning

Over time, different formulas have been developed to determine the frequencies for each sound from the diatonic scale. The Greek philosopher and mathematician Pythagoras (585-500 BCE) proposed a musical scale where the sound pitches are calculated by multiplying the perfect fifth interval (ratio 3/2). Rise to power of ratio corresponding perfect fifth musical interval will indicate the next perfect fifths frequency (Maor 2018, 16).

$\left(\frac{3}{2}\right)^{-1}$	$\left(\frac{3}{2}\right)^{0}$	$\left(\frac{3}{2}\right)^{1}$	$\left(\frac{3}{2}\right)^{2}$	$\left(\frac{3}{2}\right)^{3}$	$\left(\frac{3}{2}\right)^{4}$	$\left(\frac{3}{2}\right)^{5}$
fa	do	sol	re	la	mi	si
$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$	$\frac{81}{16}$	$\frac{243}{32}$

If for any frequency of a sound multiplied by 2 it will result in a double frequency corresponding to the upper octave of the original sound, then bringing in the same octave of the sounds will be done by multiplying or simplifying by 2. Reported to the fundamental sound in the same octave that is calculated as simple intervals, the ratio above will have the values: 4/3; 1; 3/2; 9/8; 27/16; 81/64; 243/128. Seated in order of pitch, the ratios will be: 1, 9/8; 81/64; 4/3; 3/2; 27/16; 243/128, indicating a distance between notes based on 9/8 for whole tone (1,125) and 256/243 (1,053) for halftone (Maor 2018, 16).

Intersection points

H. Helmholtz's theory (1821-1894) establishes a certain degree of affinity, a relationship between sounds based on common elements in the harmonics series. The greatest resemblance between the musical intervals of natural resonance is found between the fundamental sound and the perfect octave. In addition, the octave is not only the first and most frequently encountered of all intervals, but keeps the same ratios as any sounds in series.

3/2 = 6/4 = 12 /8…or 5/4 = 12/10

A large number of common harmonics is also found in the relationship that comes from the perfect fifth. This will indicate pairs of common overtones:

Fundamental	Harmonics				
Sol	2 (sol)	4 (sol)	6 (re)	8 (sol)	10 (si)
Do	3 (sol)	6 (sol)	9 (re)	12 (sol)	15 (si)

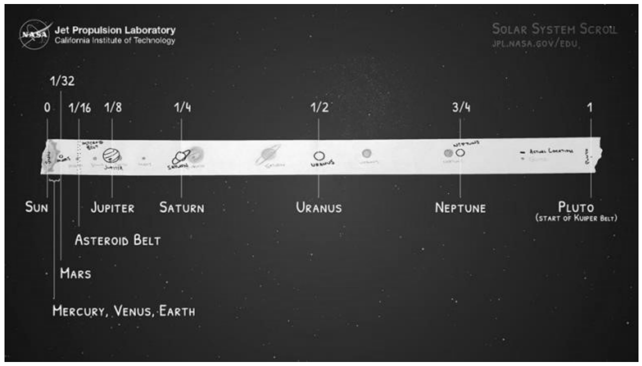
Relative to the perfect fourth, the number of common sounds diminishes. In addition, there are a series of new sounds that are not in the first sound group. Because of this, the relationship with the fourth is different, marking the beginning of the separation from the brunch of common harmonic sounds.

Fundamental	Harmonics			
Fa	3 (do)	6 (do)	9 (sol)	12 (do)
Do	4 (do)	8 (do)	12 (sol)	16 (do)

Following the same direction, we can see that the relationship between the two harmonic sounds in the major third report is further weakening. Although a number of three pairs of common sounds are kept, the appearance of foreign (different) sounds leads to a greater detachment of the links between the two fundamental sounds. Almost the same thing happens with major sixth. The minor third and the minor sixth indicate fewer affinities.

The ratio of two fundamental sounds keeps the same proportions between its harmonics. If there is 3/2 (1.5) ratio between two fundamental sounds, then the harmonic pairs will indicate the same proportion (re 9/6 and 15/10 etc.). Starting from this point of view we can say that the degree of affinity between two fundamental sounds gradually decreases, starting with the perfect octave interval, then the fifth, the fourth, the third, the sixth, and so on.This idea is the basis for organizing the degrees of affinity between sounds based on intersection zones established by affirmation of common harmonics.

The phenomenon of natural resonance in which waves are grouped into knots (intersection points) organized on the basis of accurate mathematical ratios can also be observed in relation to the distances between the planets of our solar system. For example, here is a schematic diagram of the approximate value of the planetary distances in our solar system. The ratios indicate divisions of number 2, in musical terms - perfect consonants derived from the perfect octave (octave, double octave, octave quadruple) corresponding to the ratio 1/2, 1/4, 1/8, 1/16, 1/32. If the Sun emits a fundamental sound, then most of the planets will be placed on the frequencies corresponding to the perfect octaves!



Consonance and dissonance

Since the Greek antiquity, the issue of mathematical computations in musical performances has been classified as abstract, the supreme judge being most often the aesthetic factor. That is why many definitions of consonant or dissonant feature refer to impressions, feelings, judgments based on an artistic (aesthetic) point of view rather than appealing to physical feature and mathematical ratios between sounds. However, from the earliest times, the consonant intervals were considered to be those intervals resulting from the alignment of the first harmonics, among which the perfect octave, perfect fourth, and perfect fifth occupy a privileged place. Over time, thirds and sixth entered the consonant intervals zone. Then, after the Meddle age, the musical compositions use overlapping voices leading to the gradual elimination of the fourth from the group of diatonic intervals, being treated in counterpoint and then in harmonic and polyphonic music as a dissonant interval.

Physically, in natural resonance, the consonant intervals appear before the dissonant ones, thus demonstrating a certain correspondence between the resonant phenomenon and the artistic practice in its development. However, most of the time, the assessments of the intervals, when it comes to their quality of consonant or dissonant intervals, are largely aesthetic. Professor Victor Giuleanu in his book of *Music Theory* appreciates that "in classical tonal conception, an interval is considered consonant if its component sounds, simultaneously heard, produce the impression of merging and mutual attachment”. At the same time, referring to the dissonant intervals, the author considers that "they produce the feeling of mutual rejection” (Giuleanu 1986, 212).

In Dem Urmă`s *Acoustics and Music* are presented some features of the consonant and dissonant intervals based on aesthetic assessments:

- consonant intervals – „are those combinations of sounds that produce a pleasant, equilibrium, rest, relaxation, sustained silence; consonant sounds complement each other; consonance has static tendencies, being an expected solution, a satisfactory answer.”
- dissonant intervals – „are a combination of sounds whose audition produces an unpleasant, troublesome impression, imbalance, tension, restlessness; the dissonant sounds have no affinities between them, they reject, collide; dissonance is a question waiting to be answered; it exhibits dynamic trends” (Urmă 1982, 312).

Thus, the dissonant intervals create the sensation of the need for movement, the tendency to pass to consonant intervals. To resolve a dissonance means to make it`s instability to be followed by the stability of a consonant interval. The basic principle in the compositional labor according to the rules of classical harmony is based on the alternation of dissonance-consonance, in other words on the relation: tension-relaxation. The consonant or dissonance feature of a musical interval can be identified in a melodic line and more specific way in a harmonic structure and resolving the dissonance behaves differently depending on the type of harmonic or melodic writing.

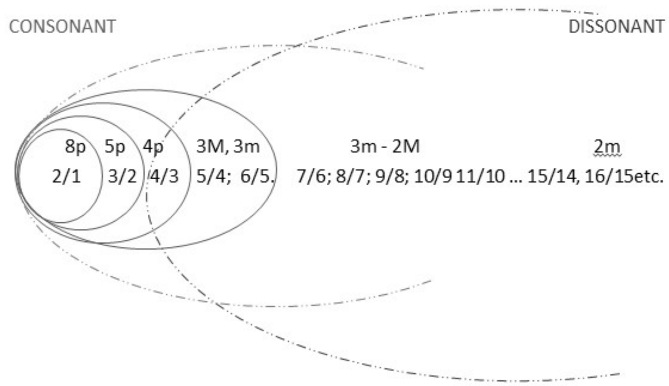
Harmonically, the dissonance is solved by leading the voices so that the tension proposed by it will be solved in the next consonant interval. In the case of the melodic movement, the memory call will make the idea of dissonance to comport differently. For example, the gradual melodic movement of a minor second is interpreted as a resolving of the tension, but it is quite different if the two sounds are produced simultaneously (harmonically), because they propose a strong dissonance, which needs to be solved in the next interval.

Gradually, since the 16th century, beginning with the development of harmonic musical writing, when composers began to pay more attention to consonance, numerous aesthetic features, attributed to certain intervals, also appear. Referring to the third, J. J. Rousseau points out in his *Musical Dictionaire* from 1767 that "the major third, which naturally urges us to joy, infuriates us when it is too large, and the minor third, the interval that urges us to gentleness, makes us sad when too tight" (Urmă 1982, 313).

The features and aesthetic quality of the intervals as well as a largely subjective approach led to the shaping of labels that will change their quality according to their time, compositional language and musical culture, some of which gradually change the place between the consonant area and the dissonant one. The evolution of the musical language has led to the gradual widening of consonance concept by assimilation of new intervals and chords and placement in this zone. These gestures initially irritated musical criticism, but they were easily accepted later.

In the musical discourse, the tension and relaxation offered by the consonance-dissonance relationship made the musical expression to develop certain internal laws, operating rules that consist in alternating contrasts. Their careful construction by preparing, expressing and solving the tensions, by carefully positioning them in the discourse with certain semantic values will lead the musical composition to a type of construction close to dramaturgy. For this reason, the analysis of the consonance-dissonance relation must be done contextually, judging carefully the role of each element in the work under analysis, considering both previous and subsequent sonorities to determine the function of the interval in the organization of the musical discourse. A separated evaluation certainly provides certain absolute values, but without place it in the concrete of an immediate sound organization. In other words, a third can be regarded as a dissonant interval in a succession of perfect fifths and octaves, but can be understood as a consonant interval in a sounds structure based on seconds and sevenths. This subjective quality valuation of the quality of the interval according to the musical context gave birth to the concept that suggests the emancipation and evolution of the idea of dissonance.

Even though the boundary between consonance and dissonance is variable, there are still areas that we can clearly identify as belonging exclusively to one or another category. For example, the perfect octave or the perfect fifth can be imagined only as a consonant interval, just as the augmented fourth or the minor second can only be seen as dissonant one. The challenge now emerging in such an approach is to set those external objectives, musical boundaries in question beyond the development of a subjective analysis to provide reasonable justification for a certain interval within the zone of consonance or dissonance. Based on these considerations, we attempted a general evaluation of the idea of consonance starting from the numerical ratios as results of the intervals within the harmonic series. Thus, we can say with certainty that the intervals that result in a relationship found in the first part of the harmonics series belong to the consonant area, and as we go further we begin to fall within the dissonant zone.



So the consonance-dissonance relationship is a linear one, which starts with the fundamental sound - indicating the maximum of consonance - then has a gradual continuation to the area of dissonance.

Conclusion

The judging of sounds in terms of consonant and dissonance also has a major aesthetic and psychological component. An absolute value of an acoustically measured range undoubtedly has some features but the way they are processed in musical practice, placing them in the discourse, preparing, executing and solving them will be a particular approach. Manipulating these tensions and relaxations will be able to reveal a construction plan with semantic elements that have a special role in the musical composition. The analyses of these sounds and the ratios between them indicate a first step in a long series of aesthetic associations beginning with musical intervals and continued with complex structures such as chords of 3, 4 or more sounds, or musical scales with specific patterns that reveal in each case a special nuance and a special ethos. So, the problem of physical sound analysis and the association of mathematical ratios of intervals provide an objective justification of sounds relation, but this judgment cannot be completed without a particular and contextual assessment of the information we receive through musical sounds.

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